"I don't want to get too caught up in the exam, but in reality I have to be faithful to anticipating what sort of questions come up"

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This paper poses some big picture questions about the notion of teaching senior secondary mathematics safely. Pre-tertiary mathematics courses typically culminate in a high stakes external examination based on core content areas such as function study, calculus, and probability and statistics. Therefore, teaching safely involves thoroughly preparing students for the examination, including teaching the full gamut of mathematical techniques prescribed by the syllabus with accuracy and clarity, with particular focus on "exam-type" problems. Is there however, more to teaching safely than teaching safely for the exam? Implicit in this question is another: Does senior secondary mathematics curricula undersell the discipline of mathematics? If so, does teaching safely mean that the teacher is strong, accomplished and creative enough to address the dilemma between curriculum pressures and teaching the discipline of mathematics? These questions are presented for exploration within the context of interview data and teaching episodes involving the voices of students and a teacher.

The following sections draw on interview excerpts and a teaching episode from a wider study, to stimulate discussion around questions relating to what it means to teach mathematics safely at the senior secondary level. 'Tom's suggestion' describes part of a lesson on applications of differential calculus.

Tom's suggestion

Mr Jones teaches an externally assessed pre-tertiary mathematics course involving an examination consisting of both questions that allow the use of a CAS-calculator (in fact these questions are designed around the use of CAS), and others that must be completed without the aid of a calculator. During a lesson on optimisation problems, Mr Jones selected the question in Figure 1 as a typical example of a non-calculator examination question.

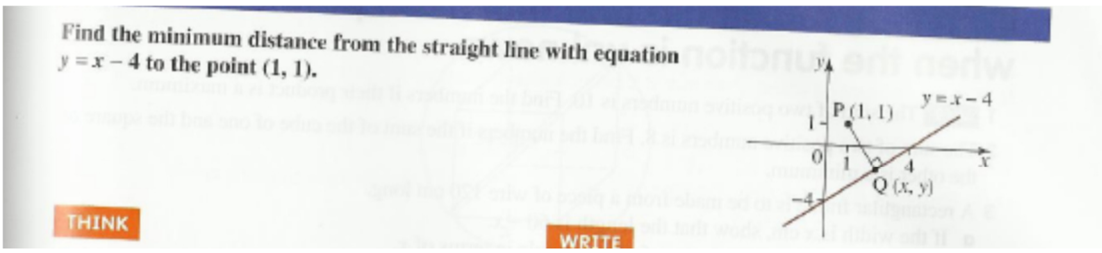

Figure 1. An example of a minimum problem when the function is not given directly (Hodgson, 2013, p. 381)

The teacher began his demonstration by reproducing the diagram shown in Figure 1. There was no mention that the minimum distance is necessarily the perpendicular distance between point P and the line $y = x - 4$. The method of solution involved finding a distance function, finding its derivative and then equating the derivative to zero to locate the x coordinate that would give the minimum distance. Mr Jones spent a few minutes reacquainting the students with the formula for the distance between two points, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, and prompted them to recognise its connection with Pythagoras Theorem. A distance function was obtained using the coordinates of points P (1,1) and Q (x, y) (see Figure 1). Emphasis was placed on the idea that the point Q(x, y) must be expressed in terms of x only, that is (x, x-4). The origin of the (x-4) was highlighted and later, in a post-lesson interview, Mr Jones commented that "often some kids don't realise when and why they need to express one variable in terms of another even if it seems quite obvious".

Once the distance function, $d(x) = \sqrt{2x^2 - 12x + 26}$ was obtained, Mr Jones reiterated that "to find the value of x which gives the minimum distance, we want the derivative $d'(x)$ to equal zero". His explanation is continued in the following excerpt from the lesson:

Mr Jones: Before I can get the derivative of this function [points to the function d(x) $= \sqrt{2x^2 - 12x + 26}$], what form do I need to put it in Jessie? It's in surd form at the moment.

Jessie: In power form.

Mr Jones: That's right power form [rewrites the function as d(x) = $(2x^2 - 12x + 26)^{\frac{1}{2}}$].

Ok so to find the derivative d'(x), what comes out the front Angela?

Angela: Umm a half.

Mr Jones: That's right $\frac{1}{2}$ and then we multiply by what Ryan?

Ryan: Oh umm the derivative of the bracket.

Mr Jones: Yes. The derivative of the bracket which is (4x – 12) and then multiplied by…what's the last bit Harry?

Harry: Umm the brackets to the power of negative a half.

Mr Jones: Yes good [completes the differentiation process to yield:

d'(x) = $\frac{1}{2}$ x (4x-12) x$(2x^2 - 12x + 26)^{-\frac{1}{2}}$]. Are we all right with that, there's your process. Ok so we've got 4x – 12 in the numerator and in the denominator we've got the 2. Remember that your negative a half [points to the expression $(2x^2 - 12x + 26)^{-\frac{1}{2}}$] moves to the denominator so we have d'(x) = $\frac{(4x-12)}{2\sqrt{2x^2-12x+26}}$. Now tell me if I've done too many steps at once there? Ok so the (4x – 2) and the 2 have stayed where they were and the bracket to the negative a half has gone underneath. Then I've just changed it from the power of a half to the square root.

The remainder of the solution process involved equating the derivative to zero to yield the value of x which gives the minimum (i.e., x = 3), and then substituting this value into the distance function to find the minimum distance.

Mr Jones: Now remember you haven't finished once you've found that x = 3 and this can be a trap for losing marks…we need to substitute back into the original function. [Following some further discussion with the class, Mr Jones records the final part of the solution on the board and concludes that the minimum distance $d(3) = \sqrt{8} = 2\sqrt{2}$].

Mr Jones: Have we needed our calculator at any stage so far? No and we don't need the calculator for this, not even to change $\sqrt{8}$ to $2\sqrt{2}$. So this question is a classic example of one that could be in the non-calculator section [of the exam]. In fact that could be a good heading for that example [writes the heading "Example of a non-calculator function unknown question].

On completion of the question a student, Tom, raised his hand and asked Mr Jones about an alternative solution method.

Tom: Mr Jones is there another way that question could be done?

Mr Jones: Tell me.

Tom: Well working out the perpendicular line, working out the intercept and then working out the distance using Pythagoras umm the distance formula.

Mr Jones: Umm… OK….say that again.

Tom: Working out the perpendicular line

Mr Jones: When you say working out the perpendicular line, what do you mean?

Tom: Yeah by using that point (1,1), and you use simultaneous equations to find that point in the middle [points to point Q shown on the diagram].

Mr Jones: OK so I know what you are saying here, so you find out the gradient of that line [points to the line $y = x - 4$].

Tom: Yeah which you can by looking at it.

Mr Jones: Yeah and so if you find the gradient of that line and then worked out the gradient of the perpendicular line and used $y - y_1 = m(x - x_1)$ you could actually do it, yes you could…umm yeah you could… ummm that would give the equation of that line but would that give us the distance though?

Tom: Yeah because you could find it using simultaneous equations. Then you can use the distance formula to find it [the distance between points P and Q].

Mr Jones: Yes, yes you could, good point. Well done.

During separate post-lesson interviews, the researcher questioned Mr Jones and Tom further about Tom's alternative solution to the minimum distance problem.

Mr Jones' interview response

R: Tom suggested an alternative method of solution for the minimum distance example. How did you decide to deal with this?

Mr Jones: Yeah well I had to be honest in one sense that the pragmatic thing is that it [the example in Figure 1] could be in the calculus section of the exam so you would have to use calculus. But what I liked about it was um you know, I mean you don't want to shut kids down, you want them to explore what the alternative is so tried not to, you know, say oh yeah good idea Tom and then move on. I wanted him to explain how his alternative method could work. The exciting thing for me is that he recognized things that we've covered before with gradient and perpendicular lines and that sort of thing. So he was taking prior knowledge umm not just from this year but from previous years and seeing it in that context which was really exciting and umm you like the fact that kids are thinking, they are not just going with the flow they are actually actively thinking, you know how else could we do this.

Tom's interview response

R: Tom you asked a question about how to solve one of the applications problems because you recognised that the two lines were at right angles to each other. Did you actually try it that way?

Tom: Yeah I did it that way first [directly using the knowledge that the two lines are perpendicular]and got the right answer and then did it the other way [using calculus].

R: Which ended up being the quickest?

Tom: Umm well my version was quicker but it can't be used like universally because it's different…

[Although Tom did not explicitly state that the slope of the path PQ is the negative reciprocal of the gradient of $y = x - 4$, this knowledge was evident in the work sample he showed the researcher during the interview].

Discussion

Mr Jones explained each step involved in using calculus to solve the minimum distance problem thoroughly. He focused on ensuring the students were fluent with procedures such as differentiating a composite function with a fractional power, such as d(x) = $(2x^2 - 12x + 26)^{\frac{1}{2}}$), and asked the class strategic questions during the solution process. Other aspects of the explanation involved making brief but explicit links between familiar concepts such as the distance formula and its connection with Pythagoras' theorem. In summary, the teaching episode transcribed in the previous section suggests that Mr Jones was teaching safely, particularly teaching safely for the exam.

Tom's alternative method of solution capitalizes on the perpendicular relationship between the line segment PQ and the line $y = x - 4$ (see Figure 1). He did not explicitly acknowledge that the shortest distance from a point to a line is the perpendicular distance but he noticed the angle between the lines was labelled a right angle in the question (see Figure 1). Mr Jones listened and responded positively to Tom's suggestion but did not pursue it further. Similarly in the interview, Mr Jones spoke favourably of Tom's approach to thinking and problem solving but he did not attend to the structure and connections between the two methods of solution, a potential source of rich discussion with the whole class. For example, using the distance function and calculus, it can be shown that the gradient of the shortest path from a point to a line is the negative reciprocal of the gradient of the line (see appendix A).

The data presented in this paper provides a context in which to situate the following thought raising questions in relation to teaching senior secondary mathematics safely:

- Is there more to teaching safely than teaching safely for the exam?
- Does senior secondary mathematics curricula undersell the discipline of mathematics?
- If so does teaching safely mean that the teacher is strong, accomplished and creative enough to address the dilemma between curriculum pressures and teaching the discipline of mathematics?
- Does teaching safely involve being aware of these kinds of teaching decisions?
- What kind of mathematical knowledge should be held by the senior mathematics teacher and how should it be held?

References

Hodgson, B. (2013). *Maths quest 12 mathematical methods CAS* (2nd ed.). Milton, Qld: John Wiley & Sons Australia.

Acknowledgement

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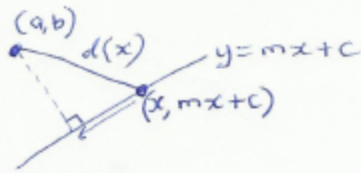
Appendix A

General solution showing that the gradient of the shortest path from a point to a line is the negative reciprocal of the gradient of the line.



$$d(x) = \sqrt{(x-a)^2 + (mx+c-b)^2}$$

$$\therefore d(x) = \left((x-a)^2 + (mx+c-b)^2 \right)^{1/2}$$

$$d'(x) = \frac{1}{2}\left(x^2 - 2ax + a^2 + m^2x^2 + 2m(c-b)x + (c-b)^2\right)^{-\frac{1}{2}} \cdot \left(2x - 2a + 2m^2x + 2m(c-b)\right)$$

$$\therefore d'(x) = \frac{(1+m^2)x + (m(c-b) - a)}{\sqrt{x^2 - 2ax + a^2 + m^2x^2 + 2m(c-b)x + (c-b)^2}}$$

Now minimum distance occurs when $d'(x) = 0$
only way $d'(x) = 0$ is when the numerator equals zero.

$$\therefore (1+m^2)x + m(c-b) - a = 0$$

$$\therefore (1+m^2)x = a - mc + mb$$

$$\therefore x = \frac{a - mc + mb}{(1+m^2)}$$

This gives the x coordinate that will give the minimum distance.

Hence $y = m\left(\frac{a-mc+mb}{1+m^2}\right) + c$

$$\therefore y = \frac{ma + m^2b + c}{1+m^2}$$

Now we know that the point $(x, mx+c)$ that results in the minimum distance is $\left(\frac{a-mc+mb}{(1+m^2)}, \frac{ma+m^2b+c}{(1+m^2)}\right)$

Hence the slope of the line connecting (a,b) and $\left(\frac{a-mc+mb}{(1+m^2)}, \frac{ma+m^2b+c}{(1+m^2)}\right)$ is:

$$\text{slope} = \left(\frac{ma+m^2b+c}{1+m^2} - b\right) \Big/ \left(\frac{a-mc+mb}{1+m^2} - a\right)$$

$$= \frac{ma+c-b}{mb-mc-am^2}$$

$$= \frac{-1}{m}$$

Now, the slope of the line $y = mx+c$ is m, we have shown that the minimum distance from a point to a line is the perpendicular to the line ($\frac{-1}{m}$ is the negative reciprocal of m).